Journal of Pure and Applied Algebra 32 (1984) 11-20
North-Holland

ULTRAPRODUCTS OF PM-RINGS AND MP-RINGS

M. CONTESSA
Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

Communicated by H. Bass
Received 27 January 1983

Introduction

It is a theorem of logic that a first-order property is preserved under the ultraproduct operation. Amitsur [1] asked if there is a direct algebraic proof of this property and he proceeded to give one. Subsequently, Bartocci [4] and Santosuosso [18] established by algebraic means that an ultraproduct of fields (integral domains, or local rings) is still a field (respectively, integral domain or local ring).

Our task in this paper is to continue in the same spirit as [1], [4], [18], showing algebraically that further properties of commutative rings with unit are preserved under the formation of ultraproducts. More precisely, the rings we shall deal with are *pm-rings* (that is, rings in which every prime ideal is contained in a unique maximal ideal) which have been extensively studied in a previous work [11] and the *mp-rings* (that is, rings in which every prime ideal contains a unique minimal prime ideal).

We investigate ultraproducts of special types of pm-rings and mp-rings. For this purpose key results are Theorem 1.4 which states that the Boolean algebra of an ultraproduct of rings is equal to the ultraproduct of the Boolean algebras of the ring factors; Theorem 1.1 which asserts that an ultraproduct of Boolean algebras is not a complete Boolean algebra if and only if the cardinalities of the Boolean algebras are unbounded modulo the ultrafilter for any ω-incomplete ultrafilter over the index set. A further result is contained in Remark 1.3, which extends Theorem 1.1 to ω-complete ultrafilters.

Furthermore, we establish the significant result that a direct product of reduced mp-rings is also an mp-ring; this fails to hold for non-reduced mp-rings.

We wish to thank Professor Karel Kripry for several helpful discussions during the preparation of this paper.

1. Notation and preliminaries

We collect in this section all notation and all the results from model theory, set theory and Boolean algebra theory that we shall need later (see [5], [8], [14] for more details). To begin with, we shall always deal with commutative rings with unit and with homomorphisms that send 1 to 1. $\mathbb{N}$ will stand for the natural numbers, and $|X|$ will denote the cardinal number of the set X. Otherwise the notation is standard.

1.1. Ultrafilters

Let I be a nonempty set. We recall that $\mathscr{P}(I)$ is the set of all subsets of I. A *filter over I* is defined to be a nonempty set $\mathscr{F}$ of $\mathscr{P}(I)$ such that

(1) $\emptyset \notin \mathscr{F}$.

(2) If $X, Y \in \mathscr{F}$, then $X \cap Y \in \mathscr{F}$.

(3) If $X \in \mathscr{F}$ and $X \subset Z \subset I$, then $Z \in \mathscr{F}$.

An *ultrafilter* $\mathscr{U}$ *over I* is a filter which is maximal in the family of filters over I (ordered by set inclusion). An ultrafilter $\mathscr{U}$ is said to be non-principal if $\{i\} \notin \mathscr{U}$ for all $i \in I$ and otherwise is principal. If I is finite, then all ultrafilters are principal, but if I is infinite, then there exist non-principal ultrafilters. A non-principal ultrafilter $\mathscr{U}$ over I is said to be *ω-complete* (or countably complete) if whenever $\{X_n \mid n<\omega\}$ is a collection of elements of $\mathscr{U}$, then $\bigcap\{X_n \mid n<\omega\}$ is also in $\mathscr{U}$. Of course $\mathscr{U}$ is *ω-incomplete* (or countably incomplete) if it is not ω-complete. This definition can be generalized to an arbitrary infinite cardinal α. The ultrafilter $\mathscr{U}$ is said to be *α-complete* if $\bigcap\{X_\xi \mid \xi<\alpha\} \in \mathscr{U}$ whenever $\xi<\alpha$ and $X_\xi \in \mathscr{U}$. If I has cardinality α, there is no non-principal α-complete ultrafilter over I; therefore all non-principal ultrafilters on a countable set are ω-incomplete.

1.2. Measurable cardinals

A cardinal α is *Ulam-measurable* if there exists on α an ω^+-complete non-principal ultrafilter. A cardinal α is *measurable* if there exists on α a β-complete non-principal ultrafilter for each $\beta<\alpha$. It has been proved that the smallest Ulam-measurable cardinal $>\omega$ is measurable (see [9]). For our purposes we shall comment that there exist ω-complete non-principal ultrafilter over I only if $|I| \geq \kappa$, where κ denotes the hypothetical first measurable cardinal.

1.3. Ultraproduct of rings

Let

$$R = \prod_{i \in I} R_i = \left\{ f : I \to \bigcup_{i \in I} R_i \mid f(i) \in R_i \right\}$$

be the direct product of the family of rings $\{R_i \mid i \in I\}$. Let $\mathscr{U}$ be an ultrafilter over I. We say that two elements f and g of R are *$\mathscr{U}$ equivalent*, in symbols $f =_{\mathscr{U}} g$, if $\{i \in I \mid f(i) = g(i)\} \in \mathscr{U}$. The binary relation $=_{\mathscr{U}}$ is a congruence relation on R. Set $\bar{R} = \prod_{i \in I} R_i / \mathscr{U}$ and let $\bar{f}$ denote the residue class of f modulo $\mathscr{U}$. The ring $\bar{R}$ is termed the *ultraproduct* of the family $\{R_i \mid i \in I\}$ with respect to the ultrafilter $\mathscr{U}$.

When the rings R_i are all equal, say to S, then $\bar{R}=S^I/\mathscr{U}$ is termed the *ultrapower* of the ring S. Moreover, since $\bar{R}\cong R_i$ for some $i\in I$ whenever the ultrafilter $\mathscr{U}$ is principal, we shall assume throughout this paper that the ultrafilters we are dealing with are non-principal; thus the index set will be infinite.

1.4. Boolean algebra associated with a ring

Let R be a ring. The set of idempotents in R will be denoted by $B(R)$ or simply by B. The set B forms a Boolean algebra provided with the following order relation: for any $e,f\in B$, $e\le f$ if $e=ef$. In this case the complement e' of e in B is $1-e$, $e\wedge f=ef$, $e\vee f=e+f-ef$, for any e,f elements of B. Also B is said to be *complete* if there exist inf and sup for any subset S of B (see [14]).

1.5. Ultraproducts of Boolean algebras

We introduce some further notation. Let I be a set and $\mathscr{U}$ be an ultrafilter on I. We shall write B_i for a Boolean algebra and $\mathscr{B}$ for $\prod_{i\in I}B_i/\mathscr{U}$, the ultraproduct. For each $i\in I$, let α_i be a cardinal number. We say that $\{\alpha_i \mid i\in I\}$ is *unbounded* modulo $\mathscr{U}$ if for each natural number n we have $\{i\in I \mid \alpha_i>n\}\in\mathscr{U}$; sometimes we will abbreviate the last statement to $\alpha_i>n$ (a.e. mod $\mathscr{U}$), where a.e. mod $\mathscr{U}$ means almost everywhere modulo $\mathscr{U}$. Of course, if the α_i are all infinite, then they are unbounded modulo $\mathscr{U}$.

Theorem 1.1. *Let I be any set and $\mathscr{U}$ a countable incomplete ultrafilter on I. Let B_i be a Boolean algebra for every $i\in I$. Then the following conditions are equivalent:*

(*i*) $\{|\mathscr{B}_i| \mid i\in I\}$ *is unbounded modulo* $\mathscr{U}$.
(ii) $\mathscr{B}=\prod_{i\in I}B_i/\mathscr{U}$ *is not a complete Boolean algebra.*

For the proof we shall need a preliminary result. First, since $\mathscr{U}$ is countably incomplete, we can pick pairwise disjoint sets I_m, where $1\le m\in\mathbb{N}$, such that $\bigcup_{m=1}^{\infty}I_m=I$ and $\bigcup_{m=1}^{n}I_m\notin\mathscr{U}$ for all $n\in\mathbb{N}$. For each $i\in I$ set $\#(i)=$the unique m such that $i\in I_m$. Also let μ_i be the number of atoms of B_i if B_i is finite and let $\mu_i=\infty$ if B_i is infinite. Finally, set $\nu_i=\min(\#(i),\mu_i)$ (hence ν_i is always finite and ≥ 1).

Lemma 1.2. *For each $j\in\mathbb{N}$, we have $\nu_i>j$* (a.e. mod $\mathscr{U}$).

Proof. We have

$$\begin{aligned}\{i\in I \mid \nu_i\le j\} &\subseteq \{i\in I \mid \#(i)\le j \text{ or } \mu_i\le j\}\\ &\subseteq \bigcup_{\kappa\le j} I_\kappa \cup \{i\in I \mid B_i \text{ has at most } j \text{ atoms}\}\\ &\subseteq \bigcup_{\kappa\le j} I_\kappa \cup \{i\in I \mid |B_i|<2^\kappa\}\end{aligned}$$

The set $\bigcup_{\kappa \le j} I_\kappa$ is not in $\mathscr{U}$ by the choice of the sets I_κ; the set $\{i \in I \mid |B_i| < 2^\kappa\}$ is also not in $\mathscr{U}$ by the assumption that $\{|\mathscr{B}_i| \mid i \in I\}$ is unbounded modulo $\mathscr{U}$. Hence $\{i \in I \mid v_i \le j\} \notin \mathscr{U}$, that is $v_i > j$ (a.e. mod $\mathscr{U}$).

Proof of Theorem 1.1. Assume that (i) holds. By the definition of v_i, we can pick for each $i \in I$ elements $b_{i,j} \in B_i$, where $j < v_i$, such that

$$b_{i,j} \ne 0 \quad \text{and} \quad b_{i,j_1} \cap b_{i,j_2} = 0 \quad \text{if } j_1 \ne j_2.$$

For each $j \in \mathbb{N}$, define

$$f_j(i) = \begin{cases} b_{i,j} & \text{if } v_i > j, \\ 1 & \text{if } v_i \le j. \end{cases}$$

Then $f_j \in \prod_{i \in I} B_i$ and $b_j := \bar{f}_j \in \mathscr{B}$. We claim that $\sup_{j \in \mathbb{N}} b_j$ does not exist in $\mathscr{B}$. Suppose the contrary and let $b = sup_{j \in \mathbb{N}} b_j$. Hence $b = \bar{f}$ for some $f \in \prod_{i \in I} B_i$. Let $l_i \in \mathbb{N}$ be maximal such that $f(i) \ge b_{i,l_i}$, and $l_i = -1$ if $f(i) \not\ge b_{i,j}$, for all j. Set

$$g(i) = \begin{cases} b_{i,l_i} & \text{if } l_i \ge 0, \\ 0 & \text{if } l_i = -1. \end{cases}$$

Then clearly $f(i) \ge g(i)$ for all $i \in I$. Hence $c := \bar{g} \le \bar{f} = b$. We prove next that $c \ne 0$. We have $\bar{f} = b \ge b_j = \bar{f}_j$, for all $j \in \mathbb{N}$. Therefore $f(i) \ge f_j(i)$ (a.e. mod $\mathscr{U}$) and for all $j \in \mathbb{N}$. But $f_j(i) = b_{i,j}$ (a.e. mod $\mathscr{U}$), as follows immediately from Lemma 1.2. Thus $f(i) \ge b_{i,j}$ (a.e. mod $\mathscr{U}$) for any fixed $j \in \mathbb{N}$. Hence

(*) $\qquad l_i \ge j$ (a.e. mod $\mathscr{U}$) for all $j \in \mathbb{N}$.

In particular $l_i \ge 0$ (a.e. mod $\mathscr{U}$); therefore

(**) $\qquad g(i) = b_{i,l_i}$ $(\ne 0)$ (a.e. mod $\mathscr{U}$).

Hence $c = \bar{g} \ne 0$ as claimed. Clearly, by (*), $l_i \ge j+1$ (a.e. mod $\mathscr{U}$). Hence

(***) $\qquad g(i) \cap b_{i,j} = b_{i,l_i} \cap b_{i,j} = 0$ (a.e. mod $\mathscr{U}$)

since $l_i \ne j$ yields $b_{i,l_i} \cap b_{i,j} = 0$ by the choice of the $b_{i,j}$ where $j < v_i$. But (***) says that

$$\bar{g} \cap \bar{f}_j = 0, \quad \text{that is,} \quad c \cap b_j = 0$$

since $f_j(i) = b_{i,j}$ (a.e. mod $\mathscr{U}$).

To summarize we have:

$$0 \ne c \le b = \sup_{j \in \mathbb{N}} b_j \quad \text{and} \quad c \cap b_j = 0 \quad \text{for all } j \in \mathbb{N},$$

which is an obvious contradiction. This completes the proof that (i)$\Rightarrow$(ii).

For the converse we shall prove that if $\{|B_i| \mid i \in I\}$ is bounded modulo $\mathscr{U}$, then B is complete (the assumption that $\mathscr{U}$ is countably incomplete is not needed for the proof of this part). Thus we are supposing that for some $n \in \mathbb{N}$, $|B_i| \le n$ (a.e.

mod $\mathscr{U}$). Set $I_j = \{i \in I \mid |B_i| = j\}$ for $j \le n$. Then $I = \bigcup_{j \le n} I_j \in \mathscr{U}$ and therefore there is some fixed j such that $I_j \in \mathscr{U}$, that is, $|B_i| = j$ (a.e. mod $\mathscr{U}$). Thus $j = 2^l$ for some l (the only possible cardinals of finite Boolean algebras being power of 2). Hence B_i is (isomorphic to) the algebra on l atoms (a.e. mod $\mathscr{U}$). Therefore we can suppose that B_i is the algebra on l atoms for all $i \in I$ (changing the algebraic structure of B_i for $i \in J \notin \mathscr{U}$ does not affect the ultraproduct). Thus let $B_i = \mathscr{A}$ be the algebra on l atoms for all $i \in I$. But then $\mathscr{B} = \prod_{i \in I} \mathscr{A}/\mathscr{U} = \mathscr{A}^I/\mathscr{U}$ is isomorphic to $\mathscr{A}$ itself; hence $\mathscr{B}$ is complete. To see that $\mathscr{B} \cong \mathscr{A}$ we proceed as follows. Let $b \in \mathscr{B}$; then $b = \bar{f}$ for some $f \in \mathscr{A}^I$ For each $a \in \mathscr{A}$, set $I_a = \{i \in I \mid f(i) = a\}$. Then $I = \bigcup_{a \in \mathscr{A}} I_a$. Since $\mathscr{A}$ is finite, $I_a \in \mathscr{U}$ for some $a \in \mathscr{A}$. Hence $f(i) = a$ (a.e. mod $\mathscr{U}$). Define $\varphi(\bar{f}) = a$. Then it is not hard to check that φ is an isomorphism of $\mathscr{B}$ onto $\mathscr{A}$.

Remark 1.3. Theorem 1.1 clarifies the situation for countably incomplete ultrafilters, that is for an index set I whose cardinality is less than the first measurable cardinal k (if it exists). When $|I| \ge k$, we have to take care of ω-complete ultrafilters as well. In this case the ultraproducts will be complete for all complete Boolean algebras whose cardinality remains bounded by a certain very large cardinal number (namely the degree of completeness of the ultrafilter). However, one has the completeness for trivial reasons – the ultraproduct is simply isomorphic to one of the factors. If the cardinalities of the algebras are allowed to approach this cardinal (or exceed it) (mod $\mathscr{U}$), the ultraproduct is again incomplete.

Another result of independent interest is the following.

Theorem 1.4. *Let I be a set and, for each $i \in I$, let R_i be a ring. Set $B_i = B(R_i)$ for all $i \in I$, and let $\mathscr{U}$ be an ultrafilter on I. Then*

$$B\left(\prod_{i \in I} R_i/\mathscr{U}\right) = \prod_{i \in I} B_i/\mathscr{U}.$$

Proof. We need only show that the set-theoretical equality holds since the Boolean operations are the same. The includion $\prod_{i \in I} B_i/\mathscr{U} \subseteq \mathscr{B}(\prod_{i \in I} R_i/\mathscr{U})$ is clear. To establish the converse, we choose any $\bar{f} \in \mathscr{B}(\prod_{i \in I} R_i/\mathscr{U})$ and set

$$H_f = \{i \in I \mid f^2(i) = f(i)\} \in \mathscr{U}.$$

We need to show that $\bar{f} = \hat{g} \in \prod_{i \in I} B_i/\mathscr{U}$ where $g \in \prod_{i \in I} B_i$. Let $g = (g(i))$ be the element of $\prod_{i \in I} R_i$ defined as follows

$$g(i) = \begin{cases} f(i) & \text{if } i \in H_f, \\ 1 & \text{if } i \notin H_f, \end{cases}$$

for all $i \in I$. Then $g \in \prod_{i \in I} B_i \subseteq \prod_{i \in I} R_i$ and $\hat{g} = \bar{f}$.

2. Ultraproducts of rings with the pm-property

We recall from [11] that a ring A is a pm-ring if and only if it satisfies the following condition: For every element $m \in A$, there exist $a, b \in A$ such that $(1-am)(1-bm')=0$, where $m'=1-m$.

Proposition 2.1. *An ultraproduct of pm-rings is a pm-ring.*

Proof. From [11] we know that a direct product of pm-rings is a pm-ring. Furthermore, passing to a quotient does not affect the order structure of prime ideals in the case of pm-rings.

Special types of pm-rings

2.1. Soft rings

Definition 2.2. A *soft* (or mou-) *ring* is a ring whose Jacobson radical is zero and whose maximal spectrum is T_2 (see [7], [11], [17]).

Proposition 2.3. *An ultraproduct of soft rings is a soft ring.*

Proof. An ultraproduct of soft rings is already a pm-ring since a soft ring is a pm-ring (see [11]), and therefore Proposition 2.1 applies. Moreover, the Jacobson radical of an ultraproduct is the ultraproduct with respect to the same ultrafilter of the Jacobson radicals (see [18]), and the proof is complete.

2.2. TB-rings

Let A be a ring and let B be the Boolean algebra associated with A. The map $\varphi : \operatorname{Max} A \to \operatorname{Spec} B$ given by $m \mapsto m \cap B$ is always continuous, surjective and closed (see [2]).

Definition 2.4. A ring A is said to be *topologically Boolean* (or, briefly, a TB-ring) if either the above map φ is injective (that is, a homeomorphism) or the following property is satisfied:

TB-property. For all $m, m' \in A$ such that $m+m'=1$, there exist $a, b, c, d, e \in A$, $e^2=e$, such that $(1-am)(1-bm')=0$, $1-am=ce$ and $1-bm'=de'$, where $e'=1-e$. (See [11].)

A TB-ring is a pm-ring and, moreover, a direct product of TB-rings is a TB-ring (see [11]).

Theorem 2.5. *The TB-property is inherited by ultraproducts.*

Proof. Given $\{A_i \mid i \in I\}$, where each A_i is a TB-ring, set $A = \prod_{i\in I} A_i$ and $B = B(A)$. It is not difficult to show that $B = \prod_{i\in I} B_i$. Let $\mathscr{U}$ be an ultrafilter on I. Set $\bar{A} = \prod_{i\in I} A_i/\mathscr{U}$ and $\bar{B} = B(\bar{A})$. By Theorem 1.4, $\bar{B} = \prod_{i\in I} B_i/\mathscr{U}$; therefore we need only show that the map α/u in the following commutative diagram is injective.

$$\begin{array}{ccc} \operatorname{Max} A & \xrightarrow{\ \sim\alpha\ } & \operatorname{Spec} B \\ \uparrow i \text{ embedding} & & \uparrow j \text{ closed embedding} \\ \operatorname{Max} \bar{A} & \xrightarrow{\ \alpha/u\ } & \operatorname{Spec} \bar{B} \end{array}$$

From the commutativity of the diagram we deduce that $(\alpha^{-1} \circ j) \circ \alpha/u = i$, which is injective; then α/u must be injective.

2.3. Complete TB-rings

Definition 2.6. A TB-ring whose Boolean algebra of idempotents is complete is said to be a *complete TB-ring.*

That a direct product of complete TB-rings is also a complete TB-ring has already been proved in [11].

Theorem 2.7. *There exist incomplete ultraproducts of complete TB-rings.*

Proof. For each integer n we take the Boolean algebra 2^n and we let $\mathscr{U}$ be any non-principal ultrafilter on $\mathbb{N}$. Then $\prod_{n\in\mathbb{N}} 2^n/\mathscr{U}$ is incomplete by Theorem 1.1.

3. Reverse topology (see [15])

Let A be a ring and let X denote the set of all prime ideals of A. Let $\operatorname{Spec} A$ be X endowed with the Zariski topology which partially orders X by set-inclusions, that is, $\mathscr{P} \le \mathscr{Q}$ if $\mathscr{P} \subseteq \mathscr{Q}$ for $\mathscr{P}, \mathscr{Q} \in X$. We give X a new topology by taking as a basis for the closed subsets the quasi-compact open subsets of $\operatorname{Spec} A$. Then X with the new topology is spectral (that is, it is the prime spectrum of a ring which can be chosen to be reduced). Moreover, the order $\preceq$ induced on X by the latter topology is precisely the reverse of the original one (that is, $\mathscr{P} \preceq \mathscr{Q}$ iff $\mathscr{Q} \le \mathscr{P}$).

Let us give to the prime spectrum of the rings introduced in Section 2 the reverse topology and let us investigate the behavior of the rings so obtained by taking direct products and ultraproducts of them. This will be done in the next section.

4. MP-property under direct product and ultraproducts

Remark 4.1. The 'duality' between the PM-property and the MP-property, which can be established by means of the reverse topology, does not extend to the topology. In fact, if a ring is pm, then its maximal spectrum is compact (that is, Hausdorff since it is always quasi-compact (see [11], [12])), but the MP-property does not imply that the minimal spectrum of the ring is compact (that is, quasi-compact since it is always Hausdorff). Take as a counter-example $C(X, \mathbb{R})$, the ring of all continuous real valued functions on a topological space X, which is an F-space but not a basically disconnected space (see [3], [13]).

For the first result of this section we need a fact from [3].

Lemma 4.2. *Let A be a reduced ring. Then the following conditions are equivalent:*
(i) *A is an mp-ring.*
(ii) *If $a, b \in A$ and $ab = 0$, then* $\mathrm{Ann}(a) + \mathrm{Ann}(b) = A$.
(iii) *For every $a, b \in A$,* $\mathrm{Ann}(a) + \mathrm{Ann}(b) = \mathrm{Ann}(ab)$.

Theorem 4.3. *A direct product (respectively, an ultraproduct) of reduced mp-rings is a reduced mp-ring.*

Proof. We can apply condition (ii) of Lemma 4.2 since it is not difficult to prove that a direct product (respectively, an ultraproduct) of reduced rings is a reduced ring as well.

Remark 4.4. If we drop the hypothesis that the rings are reduced, then Theorem 4.3 does not hold, as the following example shows.

Example 4.5. Let $A = \prod_{n=1}^{\infty} A_n$, where $A_n = \kappa[X_n, Y_n]/(X_n Y_n, X_n^n, Y_n^n)$ and let κ be any field. Set $\mathcal{T}_n = (X_n Y_n, X_n^n, Y_n^n)$. Each A_n is an Artinian local ring and hence an mp-ring with maximal ideal $\mathcal{M}_n = (x_n, y_n)$ where $x_n = X_n + \mathcal{T}_n$ and $y_n = Y_n + \mathcal{T}_n$ denote the residue classes of X_n and Y_n modulo $\mathcal{T}_n$. The element $x = (x_n)$, $x_n \in \mathcal{M}_n$, belongs to the Jacobson radical $J = \prod_{n \geq 1} \mathcal{M}_n$ of A and so does the element $y = (y_n)$, $y_n \in \mathcal{M}_n$. Moreover, x and y are not nilpotent because $x^\kappa = (x_n^\kappa) \neq 0$ and $y^\kappa = (y_n^\kappa) \neq 0$ for every integer $\kappa \geq 1$, but $xy = (x_n y_n) = 0$. Now take any maximal ideal $\mathcal{M}$ of A. There exists a minimal prime ideal $\mathcal{P} \subset \mathcal{M}$ such that $x \notin \mathcal{P}$; thus $y \in \mathcal{P}$. Also there exists a minimal prime ideal $\mathcal{Q} \subset \mathcal{M}$ such that $y \notin \mathcal{Q}$; therefore $x \in \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$. Hence A is not an mp-ring.

Special types of mp-rings

4.1. Weak Baer rings

Definition 4.6. A ring is said to be a *weak Baer ring* if the annihilator ideal of every element is principal and is generated by an idempotent (see [3], [6], [10], [19]).

A weak Baer ring is reduced and it is an mp-ring because of the following fact from [3], [6] and [10].

Theorem 4.7. *Let A be a reduced ring. Then the following conditions are equivalent:*

(i) *A is a weak Baer ring.*

(ii) *A is an mp-ring and* $\operatorname{Min} A$ *is compact.*

(iii) *A is a p.p.-ring (that is, every principal ideal is projective).*

(iv) *The ring of fractions $Q(A)$ of A is von Neumann regular (that is, reduced and 0-dimensional) and $B(Q(A))$ lies in A.*

As our first result we prove:

Theorem 4.8. *A direct product (respectively, an ultraproduct) of weak Baer rings is a weak Baer ring.*

Proof. Let $A=\prod_{\lambda\in\Lambda}A_\lambda$, where each A_λ is a weak Baer ring. Write Q_λ for the ring of fractions of A_λ. We will prove that A satisfies (iv) of Theorem 4.7. First, the isomorphism $Q(A)\xrightarrow{\sim}\prod_{\lambda\in\Lambda}Q_\lambda$ implies that $Q(A)$ is a von Neumann regular ring. Secondly, since

$$B(A)\hookrightarrow B(Q(A))\xrightarrow{\sim}B\Big(\prod_{\lambda\in\Lambda}Q_\lambda\Big)=\prod_{\lambda\in\Lambda}B(Q_\lambda)=\prod_{\lambda\in\Lambda}B(A_\lambda)=B\Big(\prod_{\lambda\in\Lambda}A_\lambda\Big)=B(A),$$

the idempotents of Q lie in A, and therefore the first part of the statement is proved. (For a different proof see [19, Lemma 2]). For the second part, let $\mathscr{U}$ be an ultrafilter over the index set Λ. Of course $\mathscr{U}$ will be non-principal. From the isomorphism $Q(\prod_{\lambda\in\Lambda}A_\lambda/\mathscr{U})\xrightarrow{\sim}\prod_{\lambda\in\Lambda}Q_\lambda/\mathscr{U}$ established in [18] it follows that $Q(\prod_{\lambda\in\Lambda}A_\lambda/\mathscr{U})$ is von Neumann regular. Moreover

$$B\Big(\prod_{\lambda\in\Lambda}A_\lambda/\mathscr{U}\Big)\hookrightarrow B\Big(Q\Big(\prod_{\lambda\in\Lambda}A_\lambda/\mathscr{U}\Big)\Big)\xrightarrow{\sim}B\Big(\prod_{\lambda\in\Lambda}Q_\lambda/\mathscr{U}\Big)=\prod_{\lambda\in\Lambda}B(Q_\lambda)/\mathscr{U}$$
$$=\prod_{\lambda\in\Lambda}B(A_\lambda)/\mathscr{U}=B\Big(\prod_{\lambda\in\Lambda}A_\lambda/\mathscr{U}\Big),$$

which concludes the proof.

4.2. *Baer rings*

Definition 4.9. A is said to be a *Baer ring* if the annihilator ideal of every ideal is principal and is generated by an idempotent (see [16]).

Of course, a Baer ring is a weak Baer ring and hence an mp-ring; however the following result makes the situation more precise.

Proposition 4.10. *A is a Baer ring if and only if it is a weak Baer ring whose Boolean algebra of idempotent elements is complete.*

Theorem 4.11. *Let* $\{A_\lambda \mid \lambda \in \Lambda\}$ *be a family of Baer rings. Set* $A = \prod_{\lambda \in \Lambda} A_\lambda$ *and* $\bar{A} = \prod_{\lambda \in \Lambda} A_\lambda / \mathcal{U}$, *where* $\mathcal{U}$ *is an ultrafilter on* Λ. *Then A is always a Baer ring, but* $\bar{A}$ *need not be.*

Proof. By virtue of Theorem 4.8 and Proposition 4.10 we only need to show that $B(A)$ (respectively, $B(\bar{A})$) is complete. That $B(A) = \prod_{\lambda \in \Lambda} B(A_\lambda)$ is complete is straightforward. By Theorem 1.4, $B(\bar{A}) = \prod_{\lambda \in \Lambda} B(A_\lambda)/\mathcal{U}$ and now we can use Theorem 1.1 and Remark 1.3 to get the required conclusion.

References

[1] S.A. Amitsur, Associative rings with identity, in: Some Aspects of Ring Theory (2° Ciclo C.I.M.E. Agosto 1965, Cremonese, 1966).

[2] K. Aoyama, On the structure space of a direct product of rings, J. Sc. Hiroshima Univ. 34 (1970) 339–353.

[3] G. Artico and U. Marconi, On the compactness of minimal spectrum, Rend. Sem. Mat. Univ. Padova 56 (1977) 79–84.

[4] U. Bartocci, Sulla generalizzazione di un teorema di Kochen, Rend. Ac. Naz. Lincei (8) 45 (1968) 221–230.

[5] J.L. Bell and A.B. Slomson, Models and Ultraproducts (North-Holland, Amsterdam, 1971).

[6] G.M. Bergman, Hereditary commutative rings and centres of hereditary rings, Proc. London Math. Soc. 213 (1971) 214–236.

[7] R. Bkouche, Couples spectraux et faisceaux associés. Applications aux anneaux de fonctions (thèse), Bull. Soc. Math. France 98 (1970) 253–295.

[8] C.C. Chang and H.J. Keisler, Model Theory (North-Holland, Amsterdam, 1977).

[9] W.W. Comfort and S. Negrepontis, The Theory of Ultrafilters (Springer, Berlin, 1974).

[10] M. Contessa, Alcune Osservazioni sugli Anelli di Baer, Ann. Univ. Ferrara XXV (1979) 219–224.

[11] M. Contessa, On PM-rings, Comm. Algebra 10 (1982) 93–108.

[12] G. De Marco and A. Orsatti, Commutative rings in which prime ideal is contained in a unique maximal ideal, Proc. Amer. Math. Soc. 30 (1971) 459–466.

[13] L. Gillman and M. Jerison, Rings of Continuous Functions (Springer, Berlin, 1976).

[14] P.R. Halmos, Lectures on Boolean Algebras (Springer, Berlin, 1974).

[15] M. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 142 (1969) 43–60.

[16] I. Kaplansky, Rings of Operators (Benjamin, New york, 1968).

[17] R.A. Levaro, Projective quasi-coherent sheaves of modules, Pacific J. Math. 57 (1975) 457–467.

[18] G. Santosuosso, Sul trasporto ad un ultraprodotto di anelli di proprietà dei suoi fattori, Rend. Math. (6) I (1968) 83–99.

[19] T.P. Speed and M.W. Evans, A note on comutative Baer Rings, J. Austral. Math. Soc. 13 (1972) 1–6.